ALGEBRA II — Lesson 0203 - Bijective Functions

DR. PAUL L. BAILEY — Thursday, February 3, 2022

1. Bijective Functions

1.1. Invertible Functions.

Definition 1. Let $f : B \to A$ be a function. An *inverse* of f is a function $g : B \to A$ such that

$$g(b) = a \quad \Leftrightarrow \quad f(a) = b.$$

That is, the inverse function reverses the effect of f. If $f(5) = 21$, then $g(21) = 5$.

We say that f is *invertible* if an inverse for g exists.

Proposition 1. *If f is invertible, its inverse is unique.*

1.2. Injective Functions.

Definition 2. Let $f : A \to B$. We say that f is *injective* (or *one-to-one*) if, for every $a_1, a_2 \in A$, we have

$$f(a_1) = f(a_2) \quad \Rightarrow \quad a_1 = a_2.$$

That is, if a_1 and a_2 are points in the domain, and if a_1 is different from a_2, then $f(a_1)$ is different from a_2. In other words, a function is one-to-one if different points in the domain get mapped to different points in the range.

For example, the function $f(x) = x^2$ is NOT one-to-one, because (for example) 2 and -2 are two different points that get mapped to the same point, namely 4.

However, the function $f(x) = x^3$ IS one-to-one, because every real number has a unique cube root.

1.3. Surjective Functions.

Definition 3. Let $f : A \to B$. We say that f is *onto* B (or *surjective*) if, for every $b \in B$, there exists $a \in A$ such that $f(a) = B$.

That is, a function is surjective if its range equals its stated codomain.

Whether or not a function is surjective depends on the stated codomain.

Example 1. We give several examples.

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. The stated codomain is $\mathbb{R}$, but the range is $[0, \infty)$.
- Let f again be given by the formula $f(x) = x^2$. But now, we state that $f : \mathbb{R} \to [0, \infty)$. Then f is onto its range, so it is surjective.
- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt[3]{x}$, the cube root of x. Then f is surjective.
- Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = 2n + 1$. Then f is not onto $\mathbb{N}$, because only odd numbers are in the range.
- Let B be the set of bees in the world, and let H be the set of beehives. Every bee has exactly one hive, so there is a function $f : B \to H$ given by $f(\text{bee})$ equals his hive. This function is surjective if and only if every hive is occupied by at least one bee.

1.4. Bijective Functions.

Definition 4. Let $f : A \to B$. We say that f is *bijective* if f is both injective and surjective.

Proposition 2. *Let $f : A \to B$. Then f is invertible if and only if f is bijective.*